Investigating snarks thru the lens of matching covered graphs

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Dedicated to beloved Professor Murty (23rd December 1940 to 13th May 2025)

It follows from the famous Vizing-Gupta (1960s) Theorem that every 2-connected cubic (that is, 3-regular) graph is either 3-edge-colorable or otherwise 4-edge-colorable; *snarks* are those that are not 3-edge-colorable, and they were named so by Martin Gardner (1976) after the mysterious and elusive object of the poem *The Hunting of the Snark* by Lewis Carroll. Snarks have received immense attention as many major open as well as solved problems — such as the Cycle Double Cover Conjecture, Tutte's 5-flow Conjecture, the Four Color Theorem, etc. — reduce to this class. For instance, it was shown by Tait (1880) that the Four Color Theorem is equivalent to the statement that there are no planar snarks. The famous Petersen graph is the smallest snark.

Observe that a cubic graph is 3-edge-colorable if and only if it has three pairwise-disjoint perfect matchings. Schönberger (1934) showed that every 2-connected cubic graph is *matching covered* — that is, each of its edges lies in some perfect matching — and one may prove this easily using Tutte's 1-factor Theorem. In particular, snarks are matching covered, and this suggests that it may be worth investigating snarks using the extensive theory of matching covered graphs.

For good reasons, most modern definitions of snarks impose an additional connectivity constraint: a 2-connected cubic graph is *essentially 4-edge-connected* if its only 3-cuts are the trivial ones (that is, all three edges incident at a vertex). Using standard matching-theoretical arguments, one may prove that every essentially 4-edge-connected cubic graph is either a *brick* (nonbipartite) or a *brace* (bipartite) — the building blocks of all matching covered graphs as per a seminal result of Lovász (1987). Consequently, all snarks are bricks.

An edge e of a brick G is *b-invariant* if the graph $G - e$ is matching covered and has precisely one brick. Carvalho, Lucchesi and Murty (`https://dl.acm.org/doi/abs/10.1006/jctb.2001.2092`) established that every brick — except K_4, the triangular prism $\overline{C_6}$ and the Petersen graph — has a b-invariant edge; thus, proving a major conjecture of Lovász. In a joint work with Carvalho, Lucchesi and Little (`https://www.combinatorics.org/ojs/index.php/eljc/article/view/v27i1p22`), we proved that each edge e of a snark G is either b-invariant or otherwise it is *quasi-b-invariant* — that is, $G - e$ is a matching covered graph with precisely two bricks. For instance, each edge of the Petersen graph is quasi-b-invariant. We also established that, except for the Petersen graph, every snark (of order n) has at most $\frac{n}{2}$ quasi-b-invariant edges.

At the recent workshop *Snarks and their generalizations*, held at Paderborn University in November 2025, Santhosh Raghul and I announced the following conjecture: the quasi-b-invariant edges of any snark, except for the Petersen graph, comprise a matching. Recently, in a joint work with Nikhil Narula, we proved this conjecture. Of course, the natural question arises: so what? Well, we don't know, and we hope to find out someday.